Switching Probability Measures

Niushan Gao

Department of Mathematics
Ryerson University
Toronto

15th Workshop on Markov Processes and Related Topics
July 2019

Objective vs Subjective Probabilities

Set up problems in an objective probabilistic world $(\Omega, \mathcal{F}, \mathbb{P})$
Work on problems in a subjective probabilistic world $(\Omega, \mathcal{F}, \mathbb{Q})$

Objective vs Subjective Probabilities

Set up problems in an objective probabilistic world $(\Omega, \mathcal{F}, \mathbb{P})$
Work on problems in a subjective probabilistic world $(\Omega, \mathcal{F}, \mathbb{Q})$

Here usually require no collapse of information:

$$\mathbb{Q} \sim \mathbb{P}.$$

Objective vs Subjective Probabilities

Set up problems in an objective probabilistic world $(\Omega, \mathcal{F}, \mathbb{P})$
Work on problems in a subjective probabilistic world $(\Omega, \mathcal{F}, \mathbb{Q})$

Here usually require no collapse of information:

$$\mathbb{Q} \sim \mathbb{P}.$$

Example (Fundamental Theorem of Asset Pricing)

If no arbitrage, then there exists $\mathbb{Q} \sim \mathbb{P}$ s.t. the discounted price process of stocks become a martingale in $(\Omega, \mathcal{F}, \mathbb{Q})$.

Objective vs Subjective Probabilities

Set up problems in an objective probabilistic world $(\Omega, \mathcal{F}, \mathbb{P})$
Work on problems in a subjective probabilistic world $(\Omega, \mathcal{F}, \mathbb{Q})$

Here usually require no collapse of information:

$$\mathbb{Q} \sim \mathbb{P}.$$

Example (Fundamental Theorem of Asset Pricing)

If no arbitrage, then there exists $\mathbb{Q} \sim \mathbb{P}$ s.t. the discounted price process of stocks become a martingale in $(\Omega, \mathcal{F}, \mathbb{Q})$.

Benefits:

$$\textit{pricing} = \textit{Expectation under } \mathbb{Q}$$

A General Question of Interest

Question

Given a set $\mathcal{K}$ in $L^0(\mathbb{P})$, can we find $\mathbb{Q} \sim \mathbb{P}$ such that $\mathcal{K}$ is "nice" in $(\Omega, \mathcal{F}, \mathbb{Q})$?

A General Question of Interest

Question

Given a set $\mathcal{K}$ in $L^0(\mathbb{P})$, can we find $\mathbb{Q} \sim \mathbb{P}$ such that $\mathcal{K}$ is "nice" in $(\Omega, \mathcal{F}, \mathbb{Q})$?

Initiated by Brannath and Schachermayer '99:

A General Question of Interest

Question

Given a set $\mathcal{K}$ in $L^0(\mathbb{P})$, can we find $\mathbb{Q} \sim \mathbb{P}$ such that $\mathcal{K}$ is "nice" in $(\Omega, \mathcal{F}, \mathbb{Q})$?

Initiated by Brannath and Schachermayer '99:

Let $\mathcal{K}$ be convex and bounded in probability, i.e., $\sup_{X \in \mathcal{K}} \mathbb{P}(|X| > n) \longrightarrow 0$ as $n \longrightarrow \infty$.

A General Question of Interest

Question

Given a set $\mathcal{K}$ in $L^0(\mathbb{P})$, can we find $\mathbb{Q} \sim \mathbb{P}$ such that $\mathcal{K}$ is "nice" in $(\Omega, \mathcal{F}, \mathbb{Q})$?

Initiated by Brannath and Schachermayer '99:

Let $\mathcal{K}$ be convex and bounded in probability, i.e., $\sup_{X \in \mathcal{K}} \mathbb{P}(|X| > n) \longrightarrow 0$ as $n \longrightarrow \infty$. Under either of the following

- $\mathcal{K}$ is solid in $L^0(\mathbb{P})$, i.e., if $X \in \mathcal{K}$ and $|Y| \leq |X|$, then $Y \in \mathcal{K}$

A General Question of Interest

Question

Given a set $\mathcal{K}$ in $L^0(\mathbb{P})$, can we find $\mathbb{Q} \sim \mathbb{P}$ such that $\mathcal{K}$ is "nice" in $(\Omega, \mathcal{F}, \mathbb{Q})$?

Initiated by Brannath and Schachermayer '99:

Let $\mathcal{K}$ be convex and bounded in probability, i.e., $\sup_{X \in \mathcal{K}} \mathbb{P}(|X| > n) \longrightarrow 0$ as $n \longrightarrow \infty$. Under either of the following

- $\mathcal{K}$ is solid in $L^0(\mathbb{P})$, i.e., if $X \in \mathcal{K}$ and $|Y| \leq |X|$, then $Y \in \mathcal{K}$
- $\mathcal{K}$ is positive (i.e., contained in $L^0_+(\mathbb{P})$)

A General Question of Interest

Question

Given a set $\mathcal{K}$ in $L^0(\mathbb{P})$, can we find $\mathbb{Q} \sim \mathbb{P}$ such that $\mathcal{K}$ is "nice" in $(\Omega, \mathcal{F}, \mathbb{Q})$?

Initiated by Brannath and Schachermayer '99:

Let $\mathcal{K}$ be convex and bounded in probability, i.e., $\sup_{X \in \mathcal{K}} \mathbb{P}(|X| > n) \longrightarrow 0$ as $n \longrightarrow \infty$. Under either of the following

- $\mathcal{K}$ is solid in $L^0(\mathbb{P})$, i.e., if $X \in \mathcal{K}$ and $|Y| \leq |X|$, then $Y \in \mathcal{K}$
- $\mathcal{K}$ is positive (i.e., contained in $L^0_+(\mathbb{P})$)

there exists $\mathbb{Q} \sim \mathbb{P}$ such that $\mathcal{K}$ is bounded in $L^1(\mathbb{Q})$.

Contributions of Kardaras et al

Question

Given a set $\mathcal{K}$ in $L^0(\mathbb{P})$, when can we find $\mathbb{Q} \sim \mathbb{P}$ such that $\mathcal{K}$ is uniformly integrable in $(\Omega, \mathcal{F}, \mathbb{Q})$?

Contributions of Kardaras et al

Question

Given a set $\mathcal{K}$ in $L^0(\mathbb{P})$, when can we find $\mathbb{Q} \sim \mathbb{P}$ such that $\mathcal{K}$ is uniformly integrable in $(\Omega, \mathcal{F}, \mathbb{Q})$?

Theorem (Kardaras and Žitković '13, Kardaras 14')

Let K be convex and bounded in $L^1(\mathbb{P})$. Suppose either

Contributions of Kardaras et al

Question

Given a set $\mathcal{K}$ in $L^0(\mathbb{P})$, when can we find $\mathbb{Q} \sim \mathbb{P}$ such that $\mathcal{K}$ is uniformly integrable in $(\Omega, \mathcal{F}, \mathbb{Q})$?

Theorem (Kardaras and Žitković '13, Kardaras 14')

Let K be convex and bounded in $L^1(\mathbb{P})$. Suppose either **(a)** *$\mathcal{K}$ is solid in $L^0(\mathbb{P})$,*

Contributions of Kardaras et al

Question

Given a set $\mathcal{K}$ in $L^0(\mathbb{P})$, when can we find $\mathbb{Q} \sim \mathbb{P}$ such that $\mathcal{K}$ is uniformly integrable in $(\Omega, \mathcal{F}, \mathbb{Q})$?

Theorem (Kardaras and Žitković '13, Kardaras 14')

Let K be convex and bounded in $L^1(\mathbb{P})$. Suppose either **(a)** *$\mathcal{K}$ is solid in $L^0(\mathbb{P})$, or* **(b)** *$\mathcal{K}$ is positive s.t. $K = \text{co}\left((X_n)_{n=1}^{\infty} \cup \{X\}\right)$ where $X_n \xrightarrow{\mathbb{P}} X$ in $\mathbb{L}^0_+(\mathbb{P})$.*

Contributions of Kardaras et al

Question

Given a set $\mathcal{K}$ in $L^0(\mathbb{P})$, when can we find $\mathbb{Q} \sim \mathbb{P}$ such that $\mathcal{K}$ is uniformly integrable in $(\Omega, \mathcal{F}, \mathbb{Q})$?

Theorem (Kardaras and Žitković '13, Kardaras 14')

Let K be convex and bounded in $L^1(\mathbb{P})$. Suppose either **(a)** *$\mathcal{K}$ is solid in $L^0(\mathbb{P})$, or* **(b)** *$\mathcal{K}$ is positive s.t. $K = \mathrm{co}\left((X_n)_{n=1}^{\infty} \cup \{X\}\right)$ where $X_n \xrightarrow{\mathbb{P}} X$ in $\mathbb{L}^0_+(\mathbb{P})$. TFAE:*

1. *$\exists \mathbb{Q} \sim \mathbb{P}$ such that the $\mathcal{K}$ is $\mathbb{Q}$-uniformly integrable*

Contributions of Kardaras et al

Question

Given a set $\mathcal{K}$ in $L^0(\mathbb{P})$, when can we find $\mathbb{Q} \sim \mathbb{P}$ such that $\mathcal{K}$ is uniformly integrable in $(\Omega, \mathcal{F}, \mathbb{Q})$?

Theorem (Kardaras and Žitković '13, Kardaras 14')

Let K be convex and bounded in $L^1(\mathbb{P})$. Suppose either **(a)** *$\mathcal{K}$ is solid in $L^0(\mathbb{P})$, or* **(b)** *$\mathcal{K}$ is positive s.t. $K = \text{co}\left((X_n)_{n=1}^{\infty} \cup \{X\}\right)$ where $X_n \xrightarrow{\mathbb{P}} X$ in $\mathbb{L}^0_+(\mathbb{P})$. TFAE:*

1. *$\exists \mathbb{Q} \sim \mathbb{P}$ such that the $\mathcal{K}$ is $\mathbb{Q}$-uniformly integrable*
2. *$\exists \mathbb{Q} \sim \mathbb{P}$ such that the $\mathbb{L}^0(\mathbb{Q})$- and $\mathbb{L}^1(\mathbb{Q})$-topologies agree on $\mathcal{K}$*

Contributions of Kardaras et al

Question

Given a set $\mathcal{K}$ in $L^0(\mathbb{P})$, when can we find $\mathbb{Q} \sim \mathbb{P}$ such that $\mathcal{K}$ is uniformly integrable in $(\Omega, \mathcal{F}, \mathbb{Q})$?

Theorem (Kardaras and Žitković '13, Kardaras 14')

Let K be convex and bounded in $L^1(\mathbb{P})$. Suppose either **(a)** *$\mathcal{K}$ is solid in $L^0(\mathbb{P})$, or* **(b)** *$\mathcal{K}$ is positive s.t. $K = \mathrm{co}\left((X_n)_{n=1}^{\infty} \cup \{X\}\right)$ where $X_n \xrightarrow{\mathbb{P}} X$ in $\mathbb{L}^0_+(\mathbb{P})$. TFAE:*

1. $\exists \mathbb{Q} \sim \mathbb{P}$ *such that the $\mathcal{K}$ is $\mathbb{Q}$-uniformly integrable*
2. $\exists \mathbb{Q} \sim \mathbb{P}$ *such that the $\mathbb{L}^0(\mathbb{Q})$- and $\mathbb{L}^1(\mathbb{Q})$-topologies agree on $\mathcal{K}$*
3. *The $\mathbb{L}^0(\mathbb{P})$-topology is locally convex on the set $\mathcal{K}$*

Contributions of Kardaras et al

Question

Given a set $\mathcal{K}$ in $L^0(\mathbb{P})$, when can we find $\mathbb{Q} \sim \mathbb{P}$ such that $\mathcal{K}$ is uniformly integrable in $(\Omega, \mathcal{F}, \mathbb{Q})$?

Theorem (Kardaras and Žitković '13, Kardaras 14')

Let K be convex and bounded in $L^1(\mathbb{P})$. Suppose either **(a)** *$\mathcal{K}$ is solid in $L^0(\mathbb{P})$, or* **(b)** *$\mathcal{K}$ is positive s.t. $K = \text{co}\left((X_n)_{n=1}^{\infty} \cup \{X\}\right)$ where $X_n \xrightarrow{\mathbb{P}} X$ in $\mathbb{L}^0_+(\mathbb{P})$. TFAE:*

1. *$\exists \mathbb{Q} \sim \mathbb{P}$ such that the $\mathcal{K}$ is $\mathbb{Q}$-uniformly integrable*
2. *$\exists \mathbb{Q} \sim \mathbb{P}$ such that the $\mathbb{L}^0(\mathbb{Q})$- and $\mathbb{L}^1(\mathbb{Q})$-topologies agree on $\mathcal{K}$*
3. *The $\mathbb{L}^0(\mathbb{P})$-topology is locally convex on the set $\mathcal{K}$*

L^0-topology $=$ convergence in probability, independent of choice of $\mathbb{Q}$.

Contributions of Kardaras et al

Question

Given a set $\mathcal{K}$ in $L^0(\mathbb{P})$, when can we find $\mathbb{Q} \sim \mathbb{P}$ such that $\mathcal{K}$ is uniformly integrable in $(\Omega, \mathcal{F}, \mathbb{Q})$?

Theorem (Kardaras and Žitković '13, Kardaras 14')

Let K be convex and bounded in $L^1(\mathbb{P})$. Suppose either **(a)** *$\mathcal{K}$ is solid in $L^0(\mathbb{P})$, or* **(b)** *$\mathcal{K}$ is positive s.t. $K = \text{co}\left((X_n)_{n=1}^{\infty} \cup \{X\}\right)$ where $X_n \xrightarrow{\mathbb{P}} X$ in $\mathbb{L}^0_+(\mathbb{P})$. TFAE:*

1. *$\exists \mathbb{Q} \sim \mathbb{P}$ such that the $\mathcal{K}$ is $\mathbb{Q}$-uniformly integrable*
2. *$\exists \mathbb{Q} \sim \mathbb{P}$ such that the $\mathbb{L}^0(\mathbb{Q})$- and $\mathbb{L}^1(\mathbb{Q})$-topologies agree on $\mathcal{K}$*
3. *The $\mathbb{L}^0(\mathbb{P})$-topology is locally convex on the set $\mathcal{K}$*

L^0-topology $=$ convergence in probability, independent of choice of $\mathbb{Q}$.

(1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are obvious for general $\mathcal{K}$.

Kardaras's Questions

Theorem (Kardaras and Žitković '13, Kardaras 14')

Let K be specified as before. TFAE:

1. $\exists \mathbb{Q} \sim \mathbb{P}$ *such that the* $\mathcal{K}$ *is* $\mathbb{Q}$*-uniformly integrable*
2. $\exists \mathbb{Q} \sim \mathbb{P}$ *such that the* $\mathbb{L}^0(\mathbb{Q})$*- and* $\mathbb{L}^1(\mathbb{Q})$*-topologies agree on* $\mathcal{K}$
3. *The* $\mathbb{L}^0(\mathbb{P})$*-topology is locally convex on the set* $\mathcal{K}$

How about the reverses for general $\mathcal{K}$?

Kardaras's Questions

Theorem (Kardaras and Žitković '13, Kardaras 14')

Let K be specified as before. TFAE:

1. $\exists \mathbb{Q} \sim \mathbb{P}$ *such that the* $\mathcal{K}$ *is* $\mathbb{Q}$*-uniformly integrable*
2. $\exists \mathbb{Q} \sim \mathbb{P}$ *such that the* $\mathbb{L}^0(\mathbb{Q})$*- and* $\mathbb{L}^1(\mathbb{Q})$*-topologies agree on* $\mathcal{K}$
3. *The* $\mathbb{L}^0(\mathbb{P})$*-topology is locally convex on the set* $\mathcal{K}$

How about the reverses for general $\mathcal{K}$? Easy: (2) $\Longrightarrow$ (1) fails.

Kardaras's Questions

Theorem (Kardaras and Žitković '13, Kardaras 14')

Let K be specified as before. TFAE:

1. $\exists \mathbb{Q} \sim \mathbb{P}$ *such that the* $\mathcal{K}$ *is* $\mathbb{Q}$*-uniformly integrable*
2. $\exists \mathbb{Q} \sim \mathbb{P}$ *such that the* $\mathbb{L}^0(\mathbb{Q})$*- and* $\mathbb{L}^1(\mathbb{Q})$*-topologies agree on* $\mathcal{K}$
3. *The* $\mathbb{L}^0(\mathbb{P})$*-topology is locally convex on the set* $\mathcal{K}$

How about the reverses for general $\mathcal{K}$? Easy: (2) $\Longrightarrow$ (1) fails.

Kardaras '14 asked:

(Q1): (2) $\Longrightarrow$ (1)if $\mathcal{K}$ is closed?
(Q2): (3) $\Longrightarrow$ (2)?

Kardaras's Questions

Theorem (Kardaras and Žitković '13, Kardaras 14')

Let K be specified as before. TFAE:

1. $\exists \mathbb{Q} \sim \mathbb{P}$ *such that the* $\mathcal{K}$ *is* $\mathbb{Q}$*-uniformly integrable*
2. $\exists \mathbb{Q} \sim \mathbb{P}$ *such that the* $\mathbb{L}^0(\mathbb{Q})$*- and* $\mathbb{L}^1(\mathbb{Q})$*-topologies agree on* $\mathcal{K}$
3. *The* $\mathbb{L}^0(\mathbb{P})$*-topology is locally convex on the set* $\mathcal{K}$

How about the reverses for general $\mathcal{K}$? Easy: (2) $\Longrightarrow$ (1) fails.

Kardaras '14 asked:

(Q1): (2) $\Longrightarrow$ (1)if $\mathcal{K}$ is closed?
(Q2): (3) $\Longrightarrow$ (2)?

Our answers: (Q1) yes! (Q2) no even if $\mathcal{K}$ is quite good.

G., D.H.Leung, F.Xanthos '19: yes to (Q1)

Main tool is "de-switching".

G., D.H.Leung, F.Xanthos '19: yes to (Q1)

Main tool is "de-switching".

Proposition

Let $\mathcal{K}$ be a convex bounded subset of $\mathbb{L}^1(\mathbb{P})$. TFAE:

1. *There exists $\mathbb{Q} \sim \mathbb{P}$ s.t. the $\mathbb{L}^0(\mathbb{Q})$- and $\mathbb{L}^1(\mathbb{Q})$-topologies agree on $\mathcal{K}$*
2. *For any $\varepsilon > 0$, $\exists A$ with $\mathbb{P}(A) > 1 - \varepsilon$ s.t.*

   $$\text{if } X_n \xrightarrow{\mathbb{P}} X \text{ in } \mathcal{K}, \text{ then } \mathbb{E}_{\mathbb{P}}\left[|X_n - X|\mathbb{1}_A\right] \longrightarrow 0$$

G., D.H.Leung, F.Xanthos '19: yes to (Q1)

Main tool is "de-switching".

Proposition

Let $\mathcal{K}$ be a convex bounded subset of $\mathbb{L}^1(\mathbb{P})$. TFAE:

1. *There exists $\mathbb{Q} \sim \mathbb{P}$ s.t. the $\mathbb{L}^0(\mathbb{Q})$- and $\mathbb{L}^1(\mathbb{Q})$-topologies agree on $\mathcal{K}$*
2. *For any $\varepsilon > 0$, $\exists A$ with $\mathbb{P}(A) > 1 - \varepsilon$ s.t.*
   if $X_n \xrightarrow{\mathbb{P}} X$ in $\mathcal{K}$, then $\mathbb{E}_{\mathbb{P}}\big[|X_n - X|\mathbb{1}_A\big] \longrightarrow 0$

$\Uparrow$ simple, based on exhaustion.

G., D.H.Leung, F.Xanthos '19: yes to (Q1)

Main tool is "de-switching".

Proposition

Let $\mathcal{K}$ be a convex bounded subset of $\mathbb{L}^1(\mathbb{P})$. TFAE:

1. *There exists $\mathbb{Q} \sim \mathbb{P}$ s.t. the $\mathbb{L}^0(\mathbb{Q})$- and $\mathbb{L}^1(\mathbb{Q})$-topologies agree on $\mathcal{K}$*
2. *For any $\varepsilon > 0$, $\exists A$ with $\mathbb{P}(A) > 1 - \varepsilon$ s.t.*

$$\text{if } X_n \xrightarrow{\mathbb{P}} X \text{ in } \mathcal{K}, \text{ then } \mathbb{E}_{\mathbb{P}}\left[|X_n - X| \mathbb{1}_A\right] \longrightarrow 0$$

$\Uparrow$ simple, based on exhaustion.

Proposition

Let $\mathcal{K}$ be a convex bounded subset of $\mathbb{L}^1(\mathbb{P})$. The following are equivalent.

1. *$\exists \mathbb{Q} \sim \mathbb{P}$ s.t. $\mathcal{K}$ is $\mathbb{Q}$-uniformly integrable*
2. *$\forall \varepsilon > 0$, $\exists A$ with $\mathbb{P}(A) > 1 - \varepsilon$ such that if $(X_n) \subset \mathcal{K}$ is Cauchy in probability, then $\mathbb{E}_{\mathbb{P}}\left[|X_n - X_m| \mathbb{1}_A\right] \longrightarrow 0$ as $n, m \longrightarrow \infty$.*

G., D.H.Leung, F.Xanthos '19: yes to (Q1)

Main tool is "de-switching".

Proposition

Let $\mathcal{K}$ be a convex bounded subset of $\mathbb{L}^1(\mathbb{P})$. TFAE:

1. *There exists $\mathbb{Q} \sim \mathbb{P}$ s.t. the $\mathbb{L}^0(\mathbb{Q})$- and $\mathbb{L}^1(\mathbb{Q})$-topologies agree on $\mathcal{K}$*
2. *For any $\varepsilon > 0$, $\exists A$ with $\mathbb{P}(A) > 1 - \varepsilon$ s.t.*

$$\text{if } X_n \xrightarrow{\mathbb{P}} X \text{ in } \mathcal{K}, \text{ then } \mathbb{E}_{\mathbb{P}}\left[|X_n - X|\mathbb{1}_A\right] \longrightarrow 0$$

$\Uparrow$ simple, based on exhaustion. based on the celebrated Komlos theorem $\downarrow$

Proposition

Let $\mathcal{K}$ be a convex bounded subset of $\mathbb{L}^1(\mathbb{P})$. The following are equivalent.

1. *$\exists \mathbb{Q} \sim \mathbb{P}$ s.t. $\mathcal{K}$ is $\mathbb{Q}$-uniformly integrable*
2. *$\forall \varepsilon > 0$, $\exists A$ with $\mathbb{P}(A) > 1 - \varepsilon$ such that if $(X_n) \subset \mathcal{K}$ is Cauchy in probability, then $\mathbb{E}_{\mathbb{P}}\left[|X_n - X_m|\mathbb{1}_A\right] \longrightarrow 0$ as $n, m \longrightarrow \infty$.*
3. *For any $\varepsilon > 0$, $\exists A$ with $\mathbb{P}(A) > 1 - \varepsilon$ s.t. $\mathcal{K}_A := \{X\mathbb{1}_A : X \in \mathcal{K}\}$ is $\mathbb{P}$-uniformly integrable.*

G., D.H.Leung, F.Xanthos '19: yes to (Q1)

Let A be as given in (2) but $\mathcal{K}_A$ is not $\mathbb{P}$-uniformly integrable.

G., D.H.Leung, F.Xanthos '19: yes to (Q1)

Let A be as given in (2) but $\mathcal{K}_A$ is not $\mathbb{P}$-uniformly integrable. Then $\exists c' > 0$ and (X_n) in $\mathcal{K}$ s.t. $\forall n \in \mathbb{N}$ and $\forall a_1, \ldots, a_n \in \mathbb{R}$,

$$\mathbb{E}_{\mathbb{P}}\left[\left|\sum_{k=1}^{n} a_k X_k \mathbb{1}_A\right|\right] \geq c' \sum_{k=1}^{n} |a_k|.$$

G., D.H.Leung, F.Xanthos '19: yes to (Q1)

Let A be as given in (2) but $\mathcal{K}_A$ is not $\mathbb{P}$-uniformly integrable. Then $\exists c' > 0$ and (X_n) in $\mathcal{K}$ s.t. $\forall n \in \mathbb{N}$ and $\forall a_1, \dots, a_n \in \mathbb{R}$,

$$\mathbb{E}_{\mathbb{P}}\left[\left|\sum_{k=1}^{n} a_k X_k \mathbb{1}_A\right|\right] \geq c' \sum_{k=1}^{n} |a_k|.$$

Applying Komlós' Theorem and relabeling, we may assume that the arithmetic means of (X_n) converge to some $X \in \mathbb{L}^0(\mathbb{P})$ a.s. Put

$$Y_n = \frac{1}{2^n} \sum_{k=1}^{2^n} X_k.$$

Clearly, $(Y_n) \subset \mathcal{K}$ is Cauchy in probability, and thus by choice of A, $\left(Y_n \mathbb{1}_A\right)$ is Cauchy in $\mathbb{L}^1(\mathbb{P})$.

G., D.H.Leung, F.Xanthos '19: yes to (Q1)

Let A be as given in (2) but $\mathcal{K}_A$ is not $\mathbb{P}$-uniformly integrable. Then $\exists c' > 0$ and (X_n) in $\mathcal{K}$ s.t. $\forall n \in \mathbb{N}$ and $\forall a_1, \dots, a_n \in \mathbb{R}$,

$$\mathbb{E}_{\mathbb{P}}\Big[\Big|\sum_{k=1}^{n} a_k X_k \mathbb{1}_A\Big|\Big] \geq c' \sum_{k=1}^{n} |a_k|.$$

Applying Komlós' Theorem and relabeling, we may assume that the arithmetic means of (X_n) converge to some $X \in \mathbb{L}^0(\mathbb{P})$ a.s. Put

$$Y_n = \frac{1}{2^n}\sum_{k=1}^{2^n} X_k.$$

Clearly, $(Y_n) \subset \mathcal{K}$ is Cauchy in probability, and thus by choice of A, $\big(Y_n \mathbb{1}_A\big)$ is Cauchy in $\mathbb{L}^1(\mathbb{P})$. On the other hand, whenever $n > m$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\big[\big|Y_n\mathbb{1}_A - Y_m\mathbb{1}_A\big|\big] &= \mathbb{E}_{\mathbb{P}}\Big[\Big|\sum_{k=1}^{2^m}\big(\frac{1}{2^n} - \frac{1}{2^m}\big)X_k\mathbb{1}_A + \sum_{k=2^m+1}^{2^n}\frac{1}{2^n}X_k\mathbb{1}_A\Big|\Big] \\
&\geq c'\Big(\sum_{k=1}^{2^m}\big(\frac{1}{2^m} - \frac{1}{2^n}\big) + \sum_{k=2^m+1}^{2^n}\frac{1}{2^n}\Big) \\
&= c'\Big(1 - \frac{2^m}{2^n} + \frac{2^n - 2^m}{2^n}\Big) \geq c'.
\end{aligned}$$

G., D.H.Leung, F.Xanthos '19: yes to (Q1)

Corollary

Let $\mathcal{K}$ be a convex bounded subset of $\mathbb{L}^1(\mathbb{P})$. TFAE:

1. *There exists $\mathbb{Q} \sim \mathbb{P}$ such that $\mathcal{K}$ is $\mathbb{Q}$-uniformly integrable*
2. *There exists $\mathbb{Q} \sim \mathbb{P}$ such that the $\mathbb{L}^0(\mathbb{Q})$- and $\mathbb{L}^1(\mathbb{Q})$-topologies agree on $\overline{\mathcal{K}}$.*

G., D.H.Leung, F.Xanthos '19: yes to (Q1)

Corollary

Let $\mathcal{K}$ be a convex bounded subset of $\mathbb{L}^1(\mathbb{P})$. TFAE:

1. *There exists $\mathbb{Q} \sim \mathbb{P}$ such that $\mathcal{K}$ is $\mathbb{Q}$-uniformly integrable*
2. *There exists $\mathbb{Q} \sim \mathbb{P}$ such that the $\mathbb{L}^0(\mathbb{Q})$- and $\mathbb{L}^1(\mathbb{Q})$-topologies agree on $\overline{\mathcal{K}}$.*

If $\mathcal{K}$ is closed, we precisely solve (Q1) in the positive.

G., D.H.Leung, F.Xanthos '19: no to (Q2)

Theorem (A)

There exists a convex bounded set $\mathcal{K}$ *in* $\mathbb{L}^1[0,1]$ *s.t.*

- $\mathbb{L}^0[0,1]$*-compact*
- *the* $\mathbb{L}^0[0,1]$*-topology on* $\mathcal{K}$ *is locally convex*

G., D.H.Leung, F.Xanthos '19: no to (Q2)

Theorem (A)

There exists a convex bounded set $\mathcal{K}$ *in* $\mathbb{L}^1[0,1]$ *s.t.*

- $\mathbb{L}^0[0,1]$*-compact*
- *the* $\mathbb{L}^0[0,1]$*-topology on* $\mathcal{K}$ *is locally convex*
- *but no* $\mathbb{Q}$*, equivalent to the Lebesgue measure on* $[0,1]$*, such that the* $\mathbb{L}^0(\mathbb{Q})$*- and* $\mathbb{L}^1(\mathbb{Q})$*-topologies agree on* $\mathcal{K}$*.*

G., D.H.Leung, F.Xanthos '19: no to (Q2)

Theorem (A)

There exists a convex bounded set $\mathcal{K}$ *in* $\mathbb{L}^1[0,1]$ *s.t.*

- $\mathbb{L}^0[0,1]$*-compact*
- *the* $\mathbb{L}^0[0,1]$*-topology on* $\mathcal{K}$ *is locally convex*
- *but no* $\mathbb{Q}$*, equivalent to the Lebesgue measure on* $[0,1]$*, such that the* $\mathbb{L}^0(\mathbb{Q})$*- and* $\mathbb{L}^1(\mathbb{Q})$*-topologies agree on* $\mathcal{K}$*.*
- *circled, i.e.,* $\mathcal{K} = -\mathcal{K}$

G., D.H.Leung, F.Xanthos '19: no to (Q2)

Theorem (A)

There exists a convex bounded set $\mathcal{K}$ in $\mathbb{L}^1[0,1]$ s.t.

- $\mathbb{L}^0[0,1]$*-compact*
- *the* $\mathbb{L}^0[0,1]$*-topology on* $\mathcal{K}$ *is locally convex*
- *but no* $\mathbb{Q}$*, equivalent to the Lebesgue measure on* $[0,1]$*, such that the* $\mathbb{L}^0(\mathbb{Q})$*- and* $\mathbb{L}^1(\mathbb{Q})$*-topologies agree on* $\mathcal{K}$*.*
- *circled, i.e.,* $\mathcal{K} = -\mathcal{K}$

Construction:

Let (X_n) be IID Cauchy rvs on $[0,1]$. For any $n \in \mathbb{N}$, put

$$Y_n = \frac{1}{\beta_n} X_n \mathbb{1}_{\{|X_n| \leq k_n\}}.$$

G., D.H.Leung, F.Xanthos '19: no to (Q2)

Theorem (A)

There exists a convex bounded set $\mathcal{K}$ in $\mathbb{L}^1[0,1]$ s.t.

- $\mathbb{L}^0[0,1]$*-compact*
- *the* $\mathbb{L}^0[0,1]$*-topology on* $\mathcal{K}$ *is locally convex*
- *but no* $\mathbb{Q}$*, equivalent to the Lebesgue measure on* $[0,1]$*, such that the* $\mathbb{L}^0(\mathbb{Q})$*- and* $\mathbb{L}^1(\mathbb{Q})$*-topologies agree on* $\mathcal{K}$*.*
- *circled, i.e.,* $\mathcal{K} = -\mathcal{K}$

Construction:

Let (X_n) be IID Cauchy rvs on $[0,1]$. For any $n \in \mathbb{N}$, put

$$Y_n = \frac{1}{\beta_n} X_n \mathbb{1}_{\{|X_n| \le k_n\}}.$$

Set

$$\mathcal{K} = \left\{ \sum_{n=1}^{\infty} a_n Y_n : \sum_{n=1}^{\infty} |a_n| \le 1 \right\}.$$

G., D.H.Leung, F.Xanthos '19: no to (Q2)

Theorem (B)

There exist a nonatomic probability space $(\Omega, \Sigma, \mathbb{P})$ *and a convex bounded set* $\mathcal{K}$ *in* $\mathbb{L}^1_+(\mathbb{P})$ *s.t.*

- *the* $\mathbb{L}^0(\mathbb{P})$*-topology on* $\mathcal{K}$ *is locally convex*
- *no* $\mathbb{Q} \sim \mathbb{P}$ *such that the* $\mathbb{L}^0(\mathbb{Q})$*- and* $\mathbb{L}^1(\mathbb{Q})$*-topologies agree on* $\mathcal{K}$

G., D.H.Leung, F.Xanthos '19: no to (Q2)

The proper topological condition is:

Theorem

Let $\mathcal{K}$ be a convex bounded subset of $\mathbb{L}^1(\mathbb{P})$. TFAE:

1. *The relative $\mathbb{L}^0(\mathbb{P})$-topology on $\mathcal{K}$ is uniformly locally convex-solid on $\mathcal{S}$.*
2. *There exists $\mathbb{Q} \sim \mathbb{P}$ s.t. the $\mathbb{L}^0(\mathbb{Q})$- and $\mathbb{L}^1(\mathbb{Q})$-topologies agree on $\mathcal{K}$.*

G., D.H.Leung, F.Xanthos '19: no to (Q2)

The proper topological condition is:

Theorem

Let $\mathcal{K}$ be a convex bounded subset of $\mathbb{L}^1(\mathbb{P})$. TFAE:

1. *The relative $\mathbb{L}^0(\mathbb{P})$-topology on $\mathcal{K}$ is uniformly locally convex-solid on $\mathcal{S}$.*
2. *There exists $\mathbb{Q} \sim \mathbb{P}$ s.t. the $\mathbb{L}^0(\mathbb{Q})$- and $\mathbb{L}^1(\mathbb{Q})$-topologies agree on $\mathcal{K}$.*

uniformly locally convex-solid on $\mathcal{K}$:
$\forall \mathbb{L}^0(\mathbb{P})$-nbhd $\mathcal{U}$ of 0, $\exists$ convex-solid $\mathcal{W} \subseteq \mathcal{U}$ such that $(X + \mathcal{W}) \cap \mathcal{K}$ is nbhd of X, for every $X \in \mathcal{K}$.

G., D.H.Leung, F.Xanthos '19: no to (Q2)

The proper topological condition is:

Theorem

Let $\mathcal{K}$ be a convex bounded subset of $\mathbb{L}^1(\mathbb{P})$. TFAE:

1. *The relative $\mathbb{L}^0(\mathbb{P})$-topology on $\mathcal{K}$ is uniformly locally convex-solid on $\mathcal{S}$.*
2. *There exists $\mathbb{Q} \sim \mathbb{P}$ s.t. the $\mathbb{L}^0(\mathbb{Q})$- and $\mathbb{L}^1(\mathbb{Q})$-topologies agree on $\mathcal{K}$.*

uniformly locally convex-solid on $\mathcal{K}$:
$\forall \mathbb{L}^0(\mathbb{P})$-nbhd $\mathcal{U}$ of 0, $\exists$ convex-solid $\mathcal{W} \subseteq \mathcal{U}$ such that $(X + \mathcal{W}) \cap \mathcal{K}$ is nbhd of X, for every $X \in \mathcal{K}$.
Compare with $(X + \mathcal{W}) \cap \mathcal{K} \subset (X + \mathcal{U}) \cap \mathcal{K}$.

G., D.H.Leung, F.Xanthos '19: no to (Q2)

The proper topological condition is:

Theorem

Let $\mathcal{K}$ be a convex bounded subset of $\mathbb{L}^1(\mathbb{P})$. TFAE:

1. *The relative $\mathbb{L}^0(\mathbb{P})$-topology on $\mathcal{K}$ is uniformly locally convex-solid on $\mathcal{S}$.*
2. *There exists $\mathbb{Q} \sim \mathbb{P}$ s.t. the $\mathbb{L}^0(\mathbb{Q})$- and $\mathbb{L}^1(\mathbb{Q})$-topologies agree on $\mathcal{K}$.*

uniformly locally convex-solid on $\mathcal{K}$:
$\forall \mathbb{L}^0(\mathbb{P})$-nbhd $\mathcal{U}$ of 0, $\exists$ convex-solid $\mathcal{W} \subseteq \mathcal{U}$ such that $(X + \mathcal{W}) \cap \mathcal{K}$ is nbhd of X, for every $X \in \mathcal{K}$.
Compare with $(X + \mathcal{W}) \cap \mathcal{K} \subset (X + \mathcal{U}) \cap \mathcal{K}$.

We improve KZ 13'.

Corollary

Let (X_n) be a bounded sequence in $\mathbb{L}^1_+(\mathbb{P})$ and let $\mathcal{K} = \mathrm{co}(X_n)$. TFAE:

1. *The $\mathbb{L}^0(\mathbb{P})$-topology is locally convex on $\mathcal{K}$.*
2. *There exists $\mathbb{Q} \sim \mathbb{P}$ s.t. the $\mathbb{L}^0(\mathbb{Q})$- and $\mathbb{L}^1(\mathbb{Q})$-topologies agree on $\mathcal{K}$.*

G., D.H.Leung, F.Xanthos '19: no to (Q2)

The main tool is the following "localized" Hahn-Banach Theorem.

Proposition

Let $\mathcal{K}$ be a convex set in $\mathbb{L}^1(\mathbb{P})$ and suppose that the relative $\mathbb{L}^0(\mathbb{P})$-topology is uniformly locally convex-solid on $\mathcal{K}$.

Then $\forall A$ with $\mathbb{P}(A) > 0$, $\exists 0 \neq Y \in \mathbb{L}^\infty_+(\mathbb{P})$, supported in A, such that

$$\mathbb{E}_\mathbb{P}\big[|X_n - X|Y\big] \longrightarrow 0 \text{ whenever } X_n \xrightarrow{\mathbb{P}} X \text{ in } \mathcal{K}$$

G., D.H.Leung, F.Xanthos '19: no to (Q2)

The main tool is the following "localized" Hahn-Banach Theorem.

Proposition

Let $\mathcal{K}$ be a convex set in $\mathbb{L}^1(\mathbb{P})$ and suppose that the relative $\mathbb{L}^0(\mathbb{P})$-topology is uniformly locally convex-solid on $\mathcal{K}$.

Then $\forall A$ with $\mathbb{P}(A) > 0$, $\exists 0 \neq Y \in \mathbb{L}^\infty_+(\mathbb{P})$, supported in A, such that

$$\mathbb{E}_{\mathbb{P}}\left[|X_n - X|Y\right] \longrightarrow 0 \text{ whenever } X_n \xrightarrow{\mathbb{P}} X \text{ in } \mathcal{K}$$

Y is viewed as linear functional on $L^1(\mathbb{P})$, continuous on $\mathcal{K}$ wrt convergence in probability.

G., D.H.Leung, F.Xanthos '19: no to (Q2)

The main tool is the following "localized" Hahn-Banach Theorem.

Proposition

Let $\mathcal{K}$ be a convex set in $\mathbb{L}^1(\mathbb{P})$ and suppose that the relative $\mathbb{L}^0(\mathbb{P})$-topology is uniformly locally convex-solid on $\mathcal{K}$.

Then $\forall A$ with $\mathbb{P}(A) > 0$, $\exists 0 \neq Y \in \mathbb{L}^\infty_+(\mathbb{P})$, supported in A, such that

$$\mathbb{E}_\mathbb{P}\big[|X_n - X|Y\big] \longrightarrow 0 \text{ whenever } X_n \xrightarrow{\mathbb{P}} X \text{ in } \mathcal{K}$$

Y is viewed as linear functional on $L^1(\mathbb{P})$, continuous on $\mathcal{K}$ wrt convergence in probability.

Above says the collection of such functionals **separates points** of $L^1(\mathbb{P})$, under given conditions.

Thanks for your attention.